

# Generalized Rényi Entropy and Structure Detection of Complex Dynamical Systems

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## Abstract

We study the problem of detecting the structure of a complex dynamical system described by a set of deterministic differential equation that contains a Hamiltonian subsystem, without any information on the explicit form of evolution laws. We suppose that initial conditions are random and the initial conditions of the Hamiltonian subsystem are independent from the initial conditions of the rest of the system. The single numerical information is the probability density function of the system at one or several, finite number of time instants. In the framework of the formalism of the generalized Rényi entropy we find necessary and sufficient conditions that the back reaction of the Hamiltonian subsystem to the rest of the system is negligible. The results can be easily generalized to the case of general, measure preserving subsystem.

## 1 Introduction

Similar generalizations of the classical Shannon entropy [1] appeared independently both in mathematical [2], [3], [4] as well as in the physical literature [5], [6], [7], [8]. These generalizations contains the same functional [9], that can be related to the Lebesgue space norm [10], [11], associated to the measure space in which the probability is defined. Observe that both generalizations, by Alfred Rényi and Constantino Tsallis, use the minimal mathematical prerequisite

necessary to define the generalized entropy: the structure of measure space, and for instance no differentiable, or further algebraic structures are assumed. Due to the similarity in their definition, both formulations generate the same probability density function (PDF), when maximal (generalized) entropy principle is used with constrained optimization. This class of generalized entropies allows to study the case of singular, normalizable PDF's, when the classical Shannon entropy is infinite [9]. The naturalness of the Rényi and Tsallis entropies, from the point of view of the category theory was proven in [12]. In the terms of physicists, this means that the functional  $N_q^{(1)}$  (see below) that appears both in the definition of the Rényi and Tsallis entropy, has nice properties. First it is multiplicative, in the case of composed system whose components are not correlated, property that translates in additivity of the Rényi entropy (RE). Secondly, the functional  $N_q^{(1)}$  is additive, in the case of composed system obtained by the measure theoretic construction known under name "direct sum", construction that appears in simplest case in [2]. In the case of PDF depending on many variables, it is possible to extend the previous generalizations of the entropy, by using only one new fact, that remains in the framework of the formalism of measure spaces: the product structure of the measure space [9]. This new generalization extends the geometrical interpretation of the RE. In this formalism the generalized Rényi entropy (GRE) is related to the norm (generalized distance) of a class of Banach spaces, a class of Lebesgue functional spaces with highly anisotropic norm [13]. It was proven that GRE has finite value also in the case of large class of PDF's, whose Shannon or Rényi entropies are infinite [9]. The GRE is additive [9] and the corresponding functional  $N_{p,q}^{(2)}$  has similar nice category theoretic properties as the functional  $N_q^{(1)}$  [12]. In the study of dynamical systems (DS) driven by external noise, modelling the anomalous transport in plasma [14], [15], the GRE plays the role of Liapunov functional [9]. In this work we expose a new application of the GRE.

There are many situations when we have little information on physical objects, for instance in astrophysics. Despite we know that part of the system is described by (classical, not quantum) Hamiltonians, or more generally, measure preserving dynamics, there are interactions that are inaccessible to observations. The situation under study is a complex dynamical system (DS)  $\Omega$ , in a finite dimensional phase space and whose dynamics is known that it is described by a set of ordinary differential equations. The DS  $\Omega$  contains the interacting subsystems,  $\Omega'$  interacting with the Hamiltonian subsystem  $\Omega''$ . The randomness, related to the entropies, appears due to random initial conditions, and we suppose that the initial conditions of the subsystems  $\Omega'$  and  $\Omega''$  are independent. In our formalism the exact form of the differential equations, of the Hamiltonian is not known. The accessible information is the probability density function on the phase space of  $\Omega$ , measured at a single or several, finite time instants. The exact formulation is contained in Subsection 2.1. In Subsection 2.2 we recall the definition of the GRE and formulate the main result contained in the Propositions 2, 4, that means by computing the RE and GRE, it is possible to decide about the existence of back reaction of the Hamiltonian subsystem  $\Omega''$

to the subsystem  $\Omega'$ . In the language of the ergodic theory, the fact that the subsystem  $\Omega''$  is driven by the subsystem  $\Omega'$ , is expressed by the statement: The DS  $\Omega$  is the skew product of the dynamical systems  $\Omega'$  with  $\Omega''$ . The proof of the results are contained in Section 3. The applicability of the previous result to the discrete time dynamical systems, when the continuous time differential equations are replaced by finite difference recursion equations is treated in the Section 4. More mathematical details are in the Appendix. We warn the reader that the proof of the Lemma 7 from the Appendix contains unproved, heuristic assumptions. Its absolutely rigorous proof (the proof the convergence of the finite dimensional approximations) requires a more elaborate mathematical framework and restrictions, that is the subject of future studies.

## 2 Statement of the problem and main result

### 2.1 Description of the dynamical system and main assumptions

Consider a composed physical system, defined in a phase space  $\Omega$ , with its subsystems  $\Omega'$  and  $\Omega''$  with their measure structures  $(\Omega', \mathcal{A}_1, m_1)$ ,  $(\Omega'', \mathcal{A}_2, m_2)$ ,  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  are the corresponding  $\sigma$ -algebras and  $m_1$ ,  $m_2$  are the corresponding measures. We develop a general formalism, that also include the case when the subsystem  $\Omega''$  is Hamiltonian. The global phase space is the direct product  $\Omega = \Omega' \times \Omega''$ , and its associated standard direct product measure space structure  $(\Omega' \times \Omega'', \mathcal{A}_1 \otimes \mathcal{A}_2, m_1 \otimes m_2)$ . A typical example is the case when  $\Omega' = \mathbb{R}^{N_1}$ ,  $\Omega'' = \mathbb{R}^{N_2}$ , the  $\sigma$ -algebras  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  are generated by open subsets from  $\mathbb{R}^{N_1}$  respectively  $\mathbb{R}^{N_2}$  and the measures  $m_1$ ,  $m_2$  are of the form

$$dm_1(\mathbf{x}) = \gamma_1(\mathbf{x}) \prod_{k=1}^{N_1} dx_k \quad (1)$$

$$dm_2(\mathbf{y}) = \gamma_2(\mathbf{y}) \prod_{k=1}^{N_2} dy_k \quad (2)$$

We consider the case when  $\Omega''$  is isomorphic to  $\mathbb{R}^{N_2}$ . Without loss of generality we can select a coordinate system such that  $\gamma_2(\mathbf{y}) \equiv 1$  and our result depends on this selection. A more general case is when  $\Omega'$  is an orientable differential manifold of dimension  $N_1$ , and in this case Eq.(1) is expressed in some local coordinate system. Consider now an evolution law on  $\Omega' \times \Omega''$ , that in a local coordinate systems is of the form

$$\frac{dx_k(t)}{dt} = U_k(t, \mathbf{x}); \quad k = \overline{1, N_1} \quad (3)$$

$$\frac{dy_j(t)}{dt} = V_j(t, \mathbf{y}, \mathbf{x}); \quad j = \overline{1, N_2} \quad (4)$$

Consider the time dependent dependent vector field on  $\Omega''$

$$\overline{V}_j^{(\mathbf{f})}(t, \mathbf{y}) := V_j(t, \mathbf{y}, \mathbf{f}(t)); \quad j = \overline{1, N_2} \quad (5)$$

where  $\mathbf{f}(t) := (f_1(t), \dots, f_{N_1}(t))$  and the associated differential equation

$$\frac{dy_j(t)}{dt} = \overline{V}_j^{(\mathbf{f})}(t, \mathbf{y}, \mathbf{f}(t)); \quad j = \overline{1, N_2} \quad (6)$$

It generate an evolution map that preserve the measure  $m_2$ , irrespective on the function  $\mathbf{f}(t) := (f_1(t), \dots, f_{N_1}(t))$ , iff:

$$\sum_{j=1}^{N_2} \frac{\partial V_j(t, \mathbf{y}, \mathbf{x})}{\partial y_j} = 0 \quad (7)$$

Observe that the important case of the Hamiltonian system in canonical variables is recovered when  $N_2 = 2d$ ,  $\mathbf{y} = (\mathbf{q}, \mathbf{p})$  and

$$\begin{aligned} V_j(t, \mathbf{y}, \mathbf{x}) &= \frac{\partial}{\partial y_{d+j}} H(t, \mathbf{y}, \mathbf{x}); \quad j = \overline{1, d} \\ V_{d+j}(t, \mathbf{y}, \mathbf{x}) &= -\frac{\partial}{\partial y_j} H(t, \mathbf{y}, \mathbf{x}); \quad j = \overline{1, d} \end{aligned}$$

Consider the situation when only the probability density function (PDF) of the initial conditions associated to the Eqs.(3, 4) is known. In this case of the random initial conditions the evolution of the joint PDF  $\rho(t, \mathbf{x}, \mathbf{y})$  is given by the following continuity equation

$$\frac{\partial \rho(t, \mathbf{x}, \mathbf{y})}{\partial t} + \frac{1}{\gamma_1(\mathbf{x})} \sum_{k=1}^{N_1} \frac{\partial}{\partial x_k} [\rho \gamma_1 U_k] + \sum_{j=1}^{N_2} \frac{\partial}{\partial y_j} [\rho V_j] = 0 \quad (8)$$

On the other hand the evolution of the subsystem  $\Omega'$  can be studied independently. The evolution of the PDF in the phase space  $\Omega'$  is described by the following continuity equation

$$\frac{\partial \rho_1(t, \mathbf{x})}{\partial t} + \frac{1}{\gamma_1(\mathbf{x})} \sum_{k=1}^{N_1} \frac{\partial}{\partial x_k} [\rho_1 \gamma_1 U_k] = 0 \quad (9)$$

Our main assumption is the following

**Condition 1** *The distribution of the random initial conditions  $\mathbf{x}(0)$ ,  $\mathbf{y}(0)$  for Eqs.(3, 4) are independent, that can be reformulated in the following condition on the initial conditions for the solution of Eq.(8), in term of the solution  $\rho_1(t, \mathbf{x})$  of Eq.(9)*

$$\rho(0, \mathbf{x}, \mathbf{y}) = \rho_1(0, \mathbf{x}) \rho_2(\mathbf{y}) \quad (10)$$

We can impose the normalization

$$\rho_2(\mathbf{y}) = \int_{\Omega_1} dm_1(\mathbf{x}) \rho(0, \mathbf{x}, \mathbf{y}) \quad (11)$$

## 2.2 The Rényi entropy, its generalization and the main results

Starting from the reinterpretation of the RE in term of distance in Lebesgue functional space, a generalization was introduced that preserve the additivity in the case of composed system without correlation. In the our formalism [9] the RE,  $S_{R,q}$  associated to the subsystem  $\Omega'$ , described by the time dependent PDF  $\rho_1(t, \mathbf{x})$  from Eq.(9) is given by

$$S_{R,q}(t; \rho_1) := \frac{1}{1-q} \log N_q^{(1)}(t; \rho_1) \quad (12)$$

where we denoted

$$N_q^{(1)}(t; \rho_1) := \int_{\Omega'} [\rho_1(t, \mathbf{x})]^q dm_1(\mathbf{x}) \quad (13)$$

In the limit  $q \rightarrow 1$  the RE is equal with the Shannon-Boltzmann entropy  $S_{classic} = - \int_{\Omega'} \rho_1(t, \mathbf{x}) \log [\rho_1(t, \mathbf{x})] dm_1(\mathbf{x})$

The solutions of Eqs.(8, 9) are related by

$$\rho_1(t, \mathbf{x}) = \int_{\Omega''} \rho(t, \mathbf{x}, \mathbf{y}) dm_2(\mathbf{y}) \quad (14)$$

The version of interest of GRE, associated to the solution of Eq.(8) is given by

$$S_{p,q}(t; \rho) := \frac{1}{1-q} \log N_{p,q}(t; \rho) \quad (15)$$

where we denoted

$$N_{p,q}^{(2)}(t; \rho) := \int_{\Omega'} dm_1(\mathbf{x}) \left[ \int_{\Omega''} dm_2(\mathbf{y}) |\rho(t, \mathbf{x}, \mathbf{y})|^q \right]^p \quad (16)$$

In the case  $p = 1$  we obtain the RE and in the limit  $q \rightarrow 1$  the limiting case of GRE, via RE, is the Shannon-Boltzmann entropy .

Define the following important functional:

$$I_{p,q}(t, \rho) := \left[ \frac{N_{p,q}^{(2)}(t; \rho)}{N_{pq}^{(1)}(t; \rho_1)} \right]^{1/p} \quad (17)$$

where Eq.(14) is assumed. We have the following

**Proposition 2** *Under the Condition 1 and previous assumptions, the functional  $I_{p,q}(t, \rho)$  associated to the solutions  $\rho(t, \mathbf{x}, \mathbf{y})$  and  $\rho_1(t, \mathbf{x})$  of the Eqs.(8, 9) has the following invariance property:*

$$I_{p,q}(t; \rho) \equiv \int_{\Omega''} dm_2(\mathbf{y}) |\rho_2(\mathbf{y})|^q \quad (18)$$

and consequently its numerical value depends only on the initial distribution function  $\rho_2(\mathbf{y})$  of the measure preserving subsystem  $\Omega''$ , and does not depend on the time  $t$ , on the parameter  $p$  as well as on the function  $\gamma_1(\mathbf{x})$  that define the measure  $dm_1(\mathbf{x})$  in Eq.(1).

**Remark 3** By using Eqs.(12, 13, 15, 16), the previous Proposition 2 can be reformulated in the term of RE of the PDF  $\rho_1, \rho_2$  and GRE of the PDF  $\rho$  as follows: the functional

$$\log I_{p,q}(t; \rho) = \frac{1}{p} [(1-q)S_{p,q}(t; \rho) - (1-pq)S_{R,pq}(t; \rho_1)] \quad (19)$$

is independent on the time  $t$ , parameter  $p$ , the choice of the measure  $dm_1(\mathbf{x})$  and has the constant value

$$\log I_{p,q}(t; \rho) = (1-q)S_{R,q}(\rho_2) \quad (20)$$

The previous Proposition 2 or its equivalent formulation Remark 3, give necessary condition for the absence of back reaction of the Hamiltonian subsystem  $\Omega''$  to the subsystem  $\Omega'$ . In the following we formulate a partial result in the reverse direction: by assuming the invariance of  $I_{p,q}$  we obtain a result about the structure of PDF similar to Eq.(32), a structure obtained assuming that there is no back reaction of the Hamiltonian subsystem  $\Omega''$  to  $\Omega'$ .

If the Condition 1 with Eq.(10) are fulfilled, then we have the following result

**Proposition 4** Suppose that the functional  $I_{p,q}(t, \rho)$  from Eq.(18) is independent on the time  $t$ , parameter  $p$  and the choice on the function  $\gamma_1(\mathbf{x})$  that define the measure  $dm_1(\mathbf{x})$ . Then there exists an map  $(t, \mathbf{x}, \mathbf{y}) \rightarrow T(t, \mathbf{x}, \mathbf{y})$ , such that for fixed  $t, \mathbf{x}$  the map  $\mathbf{y} \rightarrow T(t, \mathbf{x}, \mathbf{y})$  preserves the Lebesgue measure  $dm_2(\mathbf{y})$  and we have similar to Eq.(32)

$$\rho(t, \mathbf{x}, \mathbf{y}) = \rho_1(t, \mathbf{x}) \rho_2(T(t, \mathbf{x}, \mathbf{y})) \quad (21)$$

**Remark 5** Observe that in the case of back reaction, or equivalently, the case of completely coupled dynamical systems, the evolution from  $t = t_1$  to  $t = 0$  has the form  $(\mathbf{x}, \mathbf{y}) \rightarrow (h_1(t_1, \mathbf{x}, \mathbf{y}), h_2(t_1, \mathbf{x}, \mathbf{y}))$  and the evolution of the PDF is more complicated, compared to Eq.(21):

$$\rho(t_1, \mathbf{x}, \mathbf{y}) = \rho(0, h_1(t_1, \mathbf{x}, \mathbf{y}), h_2(t_1, \mathbf{x}, \mathbf{y})) K(t_1, \mathbf{x}, \mathbf{y}) = \quad (22)$$

$$\rho_1(0, h_1(t_1, \mathbf{x}, \mathbf{y})) \rho_2(h_2(t_1, \mathbf{x}, \mathbf{y})) K(t_1, \mathbf{x}, \mathbf{y}) \quad (23)$$

where

$$K(t_1, \mathbf{x}, \mathbf{y}) = \frac{\gamma(h_1(t_1, \mathbf{x}, \mathbf{y}), h_2(t_1, \mathbf{x}, \mathbf{y}))}{\gamma(\mathbf{x}, \mathbf{y})} \frac{\partial(h_1(t_1, \mathbf{x}, \mathbf{y}), h_2(t_1, \mathbf{x}, \mathbf{y}))}{\partial(\mathbf{x}, \mathbf{y})}$$

### 3 Proof of the results

#### 3.1 Proof of the Proposition 2

**Proof.** Denote by  $g_1^{t_1, t_0}(\mathbf{x})$  the evolution map [16], [17], [18] associated to the Eq.(3): if  $\mathbf{x}(t)$  is a solution with  $\mathbf{x}(t_0) = \mathbf{x}_0$  then  $\mathbf{x}(t) = g_1^{t, t_0}(\mathbf{x}_0)$ . Similarly we consider the evolution map  $(\mathbf{x}, \mathbf{y}) \rightarrow g^{t, t_0}(\mathbf{x}, \mathbf{y})$  associated to the system of differential equations Eqs.(3, 4), respectively let  $\mathbf{y} \rightarrow g_{\mathbf{f}}^{t, t_0}(\mathbf{y})$  the *measure preserving evolution map attached to the equations Eqs.(5, 6)*. Then the evolution of the PDF  $\rho_1$  is given by the following equation

$$\rho_1(t_1, g_1^{t_1, t_0}(\mathbf{x}_0)) \gamma_1(g_1^{t_1, t_0}(\mathbf{x}_0)) \frac{\partial g_1^{t_1, t_0}(\mathbf{x}_0)}{\partial \mathbf{x}_0} = \rho_1(t_0, \mathbf{x}_0) \gamma_1(\mathbf{x}_0) \quad (24)$$

or setting  $t_1 = 0$ ,  $\mathbf{x}_0 = \mathbf{x}$  and  $t_0 = t$ , we obtain

$$\rho_1(t, \mathbf{x}) = \frac{\gamma_1(g_1^{0, t}(\mathbf{x}))}{\gamma_1(\mathbf{x})} \frac{\partial g_1^{0, t}(\mathbf{x})}{\partial \mathbf{x}} \rho_1(0, g_1^{0, t}(\mathbf{x})) \quad (25)$$

We obtain the evolution law for the full PDF, if we decompose the map  $(\mathbf{x}, \mathbf{y}) \rightarrow g^{t, t_0}(\mathbf{x}, \mathbf{y})$  as follows

$$(\mathbf{x}, \mathbf{y}) \rightarrow g^{t_1, t_0}(\mathbf{x}, \mathbf{y}) := (g_1^{t_1, t_0}(\mathbf{x}), g_2^{t_1, t_0}(\mathbf{x}, \mathbf{y})) \quad (26)$$

where  $g_1^{t_1, t_0}$  is from Eq.(25). It follows

$$\rho(t, \mathbf{x}, \mathbf{y}) = \rho(0, g_1^{0, t}(\mathbf{x}), g_2^{0, t}(\mathbf{x}, \mathbf{y})) \frac{\gamma_1(g_1^{0, t}(\mathbf{x}))}{\gamma_1(\mathbf{x})} J(t, \mathbf{x}, \mathbf{y}) \quad (27)$$

where  $J(t, \mathbf{x}, \mathbf{y})$  is the Jacobian

$$J(t, \mathbf{x}, \mathbf{y}) = \frac{\partial (g_1^{0, t}(\mathbf{x}), g_2^{0, t}(\mathbf{x}, \mathbf{y}))}{\partial (\mathbf{x}, \mathbf{y})} = \frac{\partial (g_1^{0, t}(\mathbf{x}))}{\partial (\mathbf{x})} \frac{\partial (g_2^{0, t}(\mathbf{x}, \mathbf{y}))}{\partial (\mathbf{y})} \quad (28)$$

From the measure preserving property of the the map  $\mathbf{y} \rightarrow g_{\mathbf{f}}^{t, t_0}(\mathbf{y})$  results that for all  $\mathbf{x}$  we have (for details see the Appendix 6.1)

$$\frac{\partial (g_2^{0, t}(\mathbf{x}, \mathbf{y}))}{\partial (\mathbf{y})} = 1; \quad \forall \mathbf{x} \in \Omega' \quad (29)$$

and the Eq.(27) simplifies to the form

$$\rho(t, \mathbf{x}, \mathbf{y}) = \rho(0, g_1^{0, t}(\mathbf{x}), g_2^{0, t}(\mathbf{x}, \mathbf{y})) \frac{\gamma_1(g_1^{0, t}(\mathbf{x}))}{\gamma_1(\mathbf{x})} \frac{\partial (g_1^{0, t}(\mathbf{x}))}{\partial (\mathbf{x})} \quad (30)$$

and by using Condition 1 it follows that

$$\rho(t, \mathbf{x}, \mathbf{y}) = \rho_1(0, g_1^{0, t}(\mathbf{x})) \rho_2(g_2^{0, t}(\mathbf{x}, \mathbf{y})) \frac{\gamma_1(g_1^{0, t}(\mathbf{x}))}{\gamma_1(\mathbf{x})} \frac{\partial (g_1^{0, t}(\mathbf{x}))}{\partial (\mathbf{x})} \quad (31)$$

or by using Eqs.(25, 31) we obtain

$$\rho(t, \mathbf{x}, \mathbf{y}) = \rho_1(t, \mathbf{x}) \rho_2(g_2^{0, t}(\mathbf{x}, \mathbf{y})) \quad (32)$$

We compute now  $N_{p,q}^{(2)}(t; \rho)$  by using Eqs.(16, 32) by observing that from Eq.(29) results that for all  $\forall \mathbf{x} \in \Omega'$  and any integrable function  $F(\mathbf{y})$  we have

$$\int_{\Omega''} dm_2(\mathbf{y}) F(g_2^{0, t}(\mathbf{x}, \mathbf{y})) = \int_{\Omega''} dm_2(\mathbf{y}) F(\mathbf{y}) \quad (33)$$

Consequently by using Eqs.(16, 32) the rule Eq.(33) and the definition Eq.(13) it follows

$$N_{p,q}^{(2)}(t; \rho) = \int_{\Omega'} dm_1(\mathbf{x}) [\rho_1(t, \mathbf{x})]^{pq} \left[ \int_{\Omega''} dm_2(\mathbf{y}) [\rho_2(g_2^{0, t}(\mathbf{x}, \mathbf{y}))]^q \right]^p \quad (34)$$

$$= \int_{\Omega'} dm_1(\mathbf{x}) [\rho_1(t, \mathbf{x})]^{pq} \left[ \int_{\Omega''} dm_2(\mathbf{y}) [\rho_2(\mathbf{y})]^q \right]^p \quad (35)$$

$$= N_q^{(1)}(t; \rho_1) \left[ \int_{\Omega''} dm_2(\mathbf{y}) [\rho_2(\mathbf{y})]^q \right]^p \quad (36)$$

■

which completes the proof of Proposition 2

### 3.2 Proof of Proposition 4

Denote by  $a$  the constant value

$$a = \left[ \frac{N_{p,q}^{(2)}(t; \rho)}{N_{pq}^{(1)}(t; \rho_1)} \right]^{1/p} \quad (37)$$

By setting  $t = 0$  and from Condition 1 we obtain

$$a = \int_{\Omega''} dm_2(\mathbf{y}) |\rho_2(\mathbf{y})|^q \quad (38)$$

From Eq.(37) results

$$\int_{\Omega'} dm_1(\mathbf{x}) \left\{ a^p [\rho_1(t, \mathbf{x})]^{pq} - \left[ \int_{\Omega''} dm_2(\mathbf{y}) |\rho(t, \mathbf{x}, \mathbf{y})|^q \right]^p \right\} = 0 \quad (39)$$

Due to the independence on the measure  $dm_1$  it follows

$$a [\rho_1(t, \mathbf{x})]^q = \int_{\Omega''} dm_2(\mathbf{y}) |\rho(t, \mathbf{x}, \mathbf{y})|^q \quad (40)$$



From Eqs.(38, 40) we obtain

$$\int_{\Omega''} dm_2(\mathbf{y}) |\rho_1(t, \mathbf{x}) \rho_2(\mathbf{y})|^q = \int_{\Omega''} dm_2(\mathbf{y}) |\rho(t, \mathbf{x}, \mathbf{y})|^q \quad (41)$$

Fix for the moment the time  $t$  and the variable  $\mathbf{x}$  and we use the Lemma 7 from the Appendix, with

$$\begin{aligned} F(\mathbf{y}) &= \rho(t, \mathbf{x}, \mathbf{y}) \\ G(\mathbf{y}) &= \rho_1(t, \mathbf{x}) \rho_2(\mathbf{y}) \end{aligned}$$

Results that for all fixed  $t, \mathbf{x}$  there exists a measure preserving map

$$\Omega'' \ni \mathbf{y} \rightarrow T(t, \mathbf{x}, \mathbf{y}) \in \Omega''$$

such that

$$\rho(t, \mathbf{x}, \mathbf{y}) = \rho_1(\mathbf{x}) \rho_2(T(t, \mathbf{x}, \mathbf{y})) \quad (42)$$

that completes the proof.

## 4 Discrete time dynamical systems.

By following the arguments in the proof of Propositions 2, 4 we observe that the conclusions remain valid if the dynamical systems are described by finite difference evolution equations, instead of differential equations Eqs.(3, 4). We have the following evolutions on  $\Omega' \times \Omega''$  where again  $\Omega'' = \mathbb{R}^{N_2}$  :

$$\mathbf{x}(t + \Delta t) = \mathbf{X}(t, \mathbf{x}(t)) \quad (43)$$

$$\mathbf{y}(t + \Delta t) = \mathbf{Y}(t, \mathbf{x}(t), \mathbf{y}(t)) \quad (44)$$

where the second map preserves the Lebesgue measure ( it has unit Jacobian)

$$\frac{\partial \mathbf{Y}(t, \mathbf{x}, \mathbf{y})}{\partial \mathbf{y}} = 1$$

and in the case of discrete approximation of real physical systems the maps are also orientation preserving. In the case when the map  $\mathbf{y} \rightarrow \mathbf{Y}(t, \mathbf{x}, \mathbf{y})$  is the finite time evolution map of a Hamiltonian system, the matrix

$$\left[ \frac{\partial Y_i(t, \mathbf{x}, \mathbf{y})}{\partial y_j} \right]_{i,j=1, N_2}$$

is symplectic, but this property is not used in the proof. It is more easily to construct integrators that preserve the volume in contrast to the integrators that preserve the Poincaré invariants. If the PDF of the distribution of the initial conditions fulfill the Condition 1 and  $\rho(t, \mathbf{x}, \mathbf{y})$  is the joint PDF of the distribution at time  $t$  generated this time by maps from Eqs. Eqs.(43,44), then Propositions 2, 4 are still valid. This is important in the studies when the evolution of the DS is approximated by numerical integrators.

## 5 Conclusions.

In the case of two of interacting dynamical systems, with independent random initial conditions, when one system is Hamiltonian, it is possible to decide if there is no back reaction of the Hamiltonian system to the remaining part of the composed dynamical system. It is sufficient to compute the Rényi entropy and the generalized Rényi entropy from the joint PDF at several values of the parameters that specifies the generalized entropy, at some time instants as well as different weight function associated to the measure in the phase space. In the case of absence of back reaction the invariant defined in Eq.(17) does not depend on the parameter  $p$ , time  $t$ , the measure  $dm_1$ .

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## 6 Appendix.

### 6.1 Proof of Eq.(29)

Let  $\mathbf{x}_0 \in \Omega_1$  and denote by  $\mathbf{X}(t, \mathbf{x}_0)$  the unique solution of the Eq.(3) with the property  $\mathbf{X}(0, \mathbf{x}_0) = \mathbf{x}_0$ . Then the joint solution  $(\mathbf{x}(t), \mathbf{y}(t))$  of the Eqs.(3, 4) with the initial conditions  $(\mathbf{x}(0), \mathbf{y}(0)) = (\mathbf{x}_0, \mathbf{y}_0)$  is identical with

$$(\mathbf{X}(t, \mathbf{x}_0), \mathbf{Y}(t, \mathbf{x}_0, \mathbf{y}_0))$$

where  $\mathbf{Y}(t, \mathbf{x}_0, \mathbf{y}_0)$  is the solution of the Eqs.(5, 6) with initial condition  $\mathbf{Y}(0, \mathbf{x}_0, \mathbf{y}_0) = \mathbf{y}_0$  where we selected  $\mathbf{f}(t) \equiv \mathbf{Y}(t, \mathbf{x}_0, \mathbf{y}_0)$

$$\frac{dY_j(t, \mathbf{x}_0, \mathbf{y}_0)}{dt} = V_j(t, \mathbf{y}, \mathbf{Y}(t, \mathbf{x}_0, \mathbf{y}_0)); \quad j = \overline{1, N_2} \quad (45)$$

$$\mathbf{Y}(t, \mathbf{x}_0, \mathbf{y}_0) = \mathbf{y}_0, \quad (46)$$

Consequently the evolution map for the full system Eqs.(3, 4) is given by

$$(\mathbf{x}_0, \mathbf{y}_0) \rightarrow (\mathbf{X}(t, \mathbf{x}_0), \mathbf{Y}(t, \mathbf{x}_0, \mathbf{y}_0)) = \left( g_1^{t_1, 0}(\mathbf{x}_0), g_2^{t_1, 0}(\mathbf{x}_0, \mathbf{y}_0) \right) \quad (47)$$

Let consider  $\mathbf{x}_0$  fixed. From Eq.(7) follows that the evolution map obtained from Eq.(6), and in particular from Eqs.(45) preserves the Lebesgue measure  $dm_2(\mathbf{y})$

$$\frac{\partial (\mathbf{Y}(t, \mathbf{x}_0, \mathbf{y}_0))}{\partial (\mathbf{y}_0)} = 1; \quad \forall \mathbf{x} \in \Omega_1$$

which combined with Eq.(47) completes the proof.

## 6.2 Lemma on rearrangement

We expose a simplified proof of the following Lemma

**Lemma 7** *Suppose that the functions  $F(\mathbf{y})$ ,  $G(\mathbf{y})$  are non negative and in the complex neighborhood of  $q_0 > 0$  the functions*

$$q \rightarrow \int_{\Omega''} dm_2(\mathbf{y}) [F(\mathbf{y})]^q \quad (48)$$

$$q \rightarrow \int_{\Omega''} dm_2(\mathbf{y}) [G(\mathbf{y})]^q \quad (49)$$

*are defined, are analytic and*

$$\int_{\Omega''} dm_2(\mathbf{y}) [F(\mathbf{y})]^q = \int_{\Omega''} dm_2(\mathbf{y}) [G(\mathbf{y})]^q \quad (50)$$

*Then there exists a measurable map  $\Omega'' \ni \mathbf{y} \rightarrow T(\mathbf{y}) \in \Omega''$  such that*

$$F(\mathbf{y}) = G(T(\mathbf{y})) \quad (51)$$

**Proof.** We will approximate the map  $T(\mathbf{y})$  by constructing a sequence of maps  $\mathbf{y} \rightarrow T_n(\mathbf{y})$ . In this end first we select an increasing sequence of finite hypercube subsets  $\Omega_n \subset \Omega''$ , and in  $\Omega_n$  we select a regular covering with hypercube subdomains  $\Omega_{n,k}$ , with  $1 \leq k \leq M_n$  such that there exists a vector  $\mathbf{A}_{k,j}$  that translates  $\Omega_{n,k}$  to  $\Omega_{n,j}$  and

$$\begin{aligned} \Omega_{n,k} &\subset \Omega_n \subset \Omega'' \\ \bigcup_{k=1}^{M_n} \Omega_{n,k} &= \Omega_n \\ m_2(\Omega_{n,k}) &= \frac{m_2(\Omega_n)}{M_n} \end{aligned}$$

Denote by  $F_k = F(\mathbf{y}_k)$  respectively by  $G_k = G(\mathbf{y}_k)$  the values of the functions  $F(\mathbf{y}_k)$ ,  $G(\mathbf{y}_k)$  in some points  $\mathbf{y}_k \in \Omega_{n,k}$ . The Eq.(50) is approximated as follows

$$\sum_{k=1}^{M_n} (F_k^q - G_k^q) = 0 \quad (52)$$

From Eq.(52) we prove that exists a permutation of the indices  $k \rightarrow P_n(k)$  such that

$$F_k = G_{P_n(k)} \quad (53)$$

Without losing generality, we may assume that  $F_k > 1$ ,  $G_k > 1$ , otherwise we multiply Eq.(52) with a suitable constant  $c^q$ . Considering  $q \rightarrow \infty$  in Eq.(52) and denoting  $k_1 = \arg \max F_k$ , observe that there exists an  $k'_1$  such that  $F_{k_1} = G_{k'_1}$ . Removing this term from the sum in Eq.(52) we find the next value  $k_2 = \arg \max F_k; k \neq k_1$  and the corresponding value  $k'_2$  such that  $F_{k_2} = G_{k'_2}$ .

Continuing the process we generate a permutation  $k_j \rightarrow k'_j = P_n(k_j)$  such that  $F_{k_j} = G_{k'_j}$ . From permutation  $P_n$  we generate the map  $\mathbf{y} \rightarrow T_n(\mathbf{y})$  such that to any  $\mathbf{y} \in \Omega_{n,k}$  we associate  $T_n(\mathbf{y}) = \mathbf{y} + \mathbf{A}_{k, P_n(k)} \in \Omega_{n, P_n(k)}$ . By increasing the number of subdomains  $\Omega_{n,k}$  contained in  $\Omega_n$  and increasing  $\Omega_n$  such that  $\cup_n^\infty \Omega_n = \Omega''$  we obtain a sequence of maps  $T_n$  whose limit is the requested map  $T$ , that completes the proof. ■

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